

Electromagnetic Plane Waves (Cont'd)

Total Internal Reflection

In our discussion so far, we have assumed that $n_1^2 > n_2^2 \sin^2 \theta$. This is

always the case if $n_1 > n_2$. However, if $n_1 < n_2$, then $n_1^2 - n_2^2 \sin^2 \theta < 0$ for

$\theta > \theta_c \equiv \sin^{-1}(\frac{n_1}{n_2})$. As a result, the refraction angle is not real in this

case. The refracted wave ^{then} has the electric field of the following form:

$$\vec{E}' = \vec{E}_0 e^{ik'_1 z} e^{i(\vec{k}'_{\parallel} \cdot \vec{x} - \omega t)}$$

Where $k'_1 = \frac{i\omega}{c} \sqrt{n_2^2 \sin^2 \theta - n_1^2}$ and $k'_{\parallel} = k_{\parallel} = \frac{\omega}{c} n_1 \sin \theta$. Therefore, k'_1 is

purely imaginary, and we have:

$$\vec{E}' = \vec{E}_0 e^{-|k'_1|z} e^{i(\vec{k}'_{\parallel} \cdot \vec{x} - \omega t)}$$

This describes an inhomogeneous plane wave whose amplitude decays in a

direction that is perpendicular to the interface, while its front moves

along the interface. We note that:

$$\vec{\nabla} \cdot \vec{E}' = 0 \Rightarrow \vec{k}' \cdot \vec{E}' = 0 \Rightarrow k'_{11} \cdot \vec{E}'_{011} + i |k'_z| E'_{0z} = 0$$

Hence, E'_{0z} and $k'_{11} \cdot \vec{E}'_{011}$ have a $\frac{\pi}{2}$ phase difference. The energy flow

normal to the interface is given by (after time-averaging):

$$S' = \frac{1}{2} \text{Re}(\vec{E}' \times \vec{H}'^*) \cdot \hat{z} = \frac{1}{2} \text{Re} \left[\vec{E}' \times \frac{(\vec{k}'^* \times \vec{E}'^*)}{\omega \mu'} \right] \cdot \hat{z} = \frac{1}{2 \omega \mu'} \text{Re} \left[(\vec{E}' \cdot \vec{E}'^*) (\vec{k}'^* \cdot \hat{z}) - (\vec{k}'^* \cdot \vec{E}') (\vec{E}'^* \cdot \hat{z}) \right]$$

The first term inside the bracket is purely imaginary, thus:

$$S' = -\frac{1}{2 \omega \mu'} \text{Re} \left[(\vec{k}'^* \cdot \vec{E}') E'_z{}^* \right]$$

For "s polarization", $E'_z = 0$ implying that $S' = 0$. For "p polarization",

however, $E'_z \neq 0$. In this case, we can use:

$$\vec{k}' \cdot \vec{E}' = 0 \Rightarrow k'_z E'_z + k'_{11} \cdot \vec{E}'_{11} = 0 \Rightarrow k'_{11} \cdot \vec{E}'_{11} = -k'_z E'_z$$

\uparrow \uparrow
 $\vec{\nabla} \cdot \vec{E}' = 0$ \vec{k}'_{11} is real

This leads to:

$$S' = -\frac{1}{2 \omega \mu'} \text{Re} \left[(\vec{k}'^* \cdot \vec{E}') E'_z{}^* \right] = -\frac{1}{2 \omega \mu'} \text{Re} \left[-2i |k'_z| E'_z \right] = 0$$

Therefore, as expected, there is energy flow normal to the interface

when we have total internal reflection.

Multiple Parallel Interfaces

Considering N media, we have

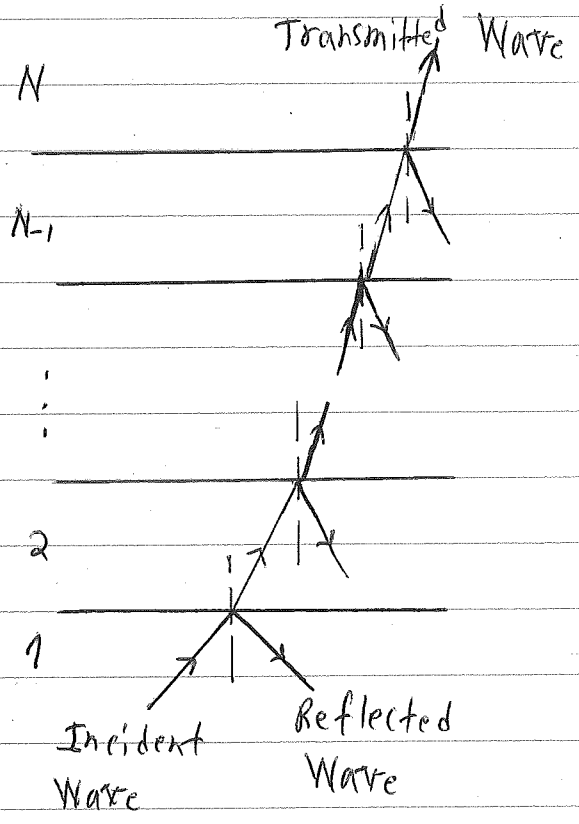
$N-2$ internal layers and $N-1$

interfaces. In each layer, and the first medium,

two plane wave exist. While, in the

final medium there is only the

transmitted wave.

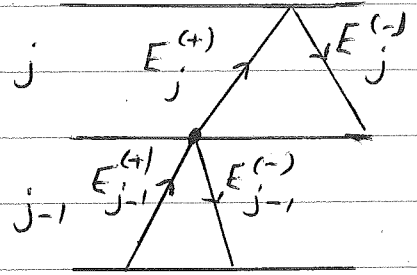


Let us consider the interface that separates layers $j-1$ and j :

We have:

$$E_j^{(+)} = t_{j-1,j} E_{j-1}^{(+)} + r_{j,j-1} E_j^{(-)}$$

$$E_{j-1}^{(-)} = r_{j-1,j} E_{j-1}^{(+)} + t_{j,j-1} E_j^{(-)}$$



Here, t_{nm} is the amplitude transmission coefficient from layer "n" to layer "m", and r_{nm} is the amplitude reflection coefficient from

layer "n" to layer "m". The quantities $E_j^{(+)}, E_j^{(-)}, E_{j-1}^{(+)}, E_{j-1}^{(-)}$ are

wave

the Λ values at the interface between the layers j and $j-1$. After

using the relations $r_{j,j-1} = -r_{j-1,j}$ and $r_{j,j-1}^2 + t_{j,j-1}t_{j-1,j} = 1$, we can

write $E_j^{(\pm)}$ in terms of $E_{j-1}^{(\pm)}$ as follows:

$$\begin{bmatrix} E_j^{(+)} \\ E_j^{(-)} \end{bmatrix} = \begin{bmatrix} 1 & r_{j,j-1} \\ t_{j,j-1} & t_{j,j-1} \\ r_{j,j-1} & 1 \\ t_{j,j-1} & t_{j,j-1} \end{bmatrix} \begin{bmatrix} E_{j-1}^{(+)} \\ E_{j-1}^{(-)} \end{bmatrix}$$

The 2×2 matrix, denoted by $\Pi^{(j)}$ is called the transfer matrix of the j -th interface.

We also need to take propagation in layer j into account. This is done by considering the phase difference at the top and bottom of that layer:

$$\begin{bmatrix} E_j^{(+)} \text{ (top)} \\ E_j^{(-)} \text{ (top)} \end{bmatrix} = \underbrace{\begin{bmatrix} e^{ik_z^{(j)} d_j} & 0 \\ 0 & e^{-ik_z^{(j)} d_j} \end{bmatrix}}_{P^{(j)}} \begin{bmatrix} E_j^{(+)} \text{ (bottom)} \\ E_j^{(-)} \text{ (bottom)} \end{bmatrix}$$

Here, d_j is the thickness of the j -th layer.

We can then write:

$$\begin{bmatrix} E_N^{(+)} \\ E_N^{(-)} \end{bmatrix} = \Pi^{(N-2)} P^{(N-2)} \Pi^{(N-3)} P^{(N-3)} \dots \Pi^{(0)} \begin{bmatrix} E_1^{(+)} \\ E_1^{(-)} \end{bmatrix}$$

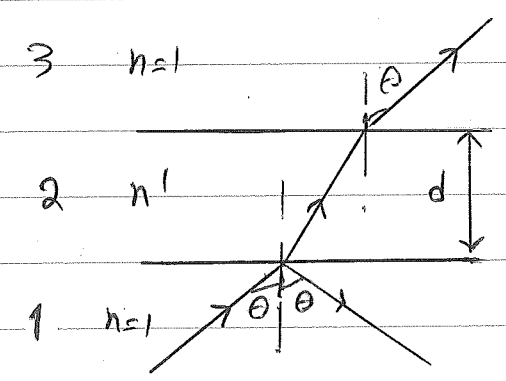
Here $E_N^{(+)}$ is the transmitted wave, $E_N^{(-)} = 0$, $E_1^{(-)}$ is the reflected wave, and $E_1^{(+)}$ is the incident wave. This leaves us with two equations for two unknowns ($E_1^{(-)}$ and $E_N^{(+)}$).

We note that in above we have assumed "s polarization" or "p polarization" with the corresponding r_{nm} and t_{nm} coefficients. A general polarization can be written as a superposition of these polarizations and treated accordingly.

Example: A simple dielectric layer, polarization perpendicular to to the plane of incidence.

In this case, we have:

$$r_{12} = \frac{n \cos \theta - n' \cos \theta'}{n \cos \theta + n' \cos \theta'} = -r_{21}$$



$$t_{12} = \frac{2n \cos \theta}{n \cos \theta + n' \cos \theta'} \quad , \quad t_{21} = \frac{2n' \cos \theta'}{n \cos \theta + n' \cos \theta'}$$

Also:

$$r_{23} = r_{21} \quad , \quad t_{23} = t_{32} \quad , \quad r_{32} = r_{12} \quad , \quad t_{32} = t_{12}$$

Then:

$$\mathbb{I}^{(0)} = \begin{bmatrix} 1 & r_{21} \\ t_{21} & t_{21} \\ r_{21} & 1 \\ t_{21} & t_{21} \end{bmatrix} \quad , \quad \mathbb{I}^{(1)} = \begin{bmatrix} 1 & r_{12} \\ t_{12} & t_{12} \\ r_{12} & 1 \\ t_{12} & t_{12} \end{bmatrix} \quad , \quad \mathbb{P}^{(1)} = \begin{bmatrix} e^{i n' k \cos \theta' d} & 0 \\ 0 & e^{-i n' k \cos \theta' d} \end{bmatrix}$$

Thus:

$$\begin{bmatrix} E_3^{(+)} \\ 0 \end{bmatrix} = \underbrace{\mathbb{I}^{(1)} \mathbb{P}^{(1)} \mathbb{I}^{(0)}}_{\mathbb{\Pi}} \begin{bmatrix} E_1^{(+)} \\ r E_1^{(+)} \end{bmatrix}$$

$$\mathbb{\Pi} = \begin{bmatrix} e^{i d} + e^{-i d} \frac{r_{12} r_{21}}{t_{12} t_{21}} & e^{-i d} \frac{r_{12}}{t_{12} t_{21}} \\ e^{i d} \frac{r_{21}}{t_{12} t_{21}} + e^{-i d} \frac{r_{12}}{t_{12} t_{21}} & e^{-i d} \frac{r_{12}}{t_{12} t_{21}} \end{bmatrix}$$

Where: $(\alpha \equiv n' k \cos \theta' d)$

$$r = \frac{2i r_{12} \sinh \alpha}{(e^{-i \alpha} - e^{i \alpha} r_{12})}$$

It is seen that $r \neq 0$ if $\alpha = m\pi$ (m an integer), hence no

reflection. In this case the incident wave is completely transmitted.

The condition for this to happen is:

$$2d n' \cos \theta' = m \lambda \Rightarrow d \cos \theta' = \frac{m \lambda}{2 n'}$$

This phenomenon, called "transmission" or "Fabry-Perot resonance",

can be understood as a result of destructive interference

between the directly reflected wave at the 1-2 interface and

the wave refracted at that interface then reflected at the

2-3 interface and then refracted back to 1.